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The motion of a spherical liquid drop at high Reynolds number

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The steady motion of a liquid drop in another liquid of comparable density and viscosity is studied theoretically. Both inside and outside the drop, the Reynolds number is taken to be large enough for boundary-layer theory to hold, but small enough for surface tension to keep the drop nearly spherical. Surface-active impurities are assumed absent. We investigate the boundary layers associated with the inviscid first approximation to the flow, which is shown to be Hill's spherical vortex inside, and potential flow outside. The boundary layers are shown to perturb the velocity field only slightly at high Reynolds numbers, and to obey linear equations which are used to find first and second approximations to the drag coefficient and the rate of internal circulation.

Drag coefficients calculated from the theory agree quite well with experimental values for liquids which satisfy the conditions of the theory. There appear to be no experimental results available to test our prediction of the internal circulation.

1. Introduction

Let us consider the steady motion, under gravity for example, of a drop of one liquid in another immiscible liquid of comparable density and viscosity, when the Reynolds numbers of both the internal circulation and the exterior flow are large. We assume that the surface tension pressure is very much larger than either the exterior or the interior dynamic pressure, so that the drop is nearly spherical. Finally, we assume that the interface is free from surface-active agents so that surface tension gradients are negligible (see Harper, Moore & Pearson 1967).

These assumptions simplify the problem enough to permit some progress with an analytical solution of the equations of motion without preventing flows satisfying them from being experimentally realized (see §9). The absence of any rigid surface in the flow means that at sufficiently large Reynolds numbers the velocity is nowhere very different from what it would be in an ideal fluid. This observation was first made by Levich (1949) for the special case of a gas bubble, for which the ideal flow is just potential flow past a sphere. However, the surface of the

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bubble must be tangentially stress-free; i.e. the tangential viscous stress component must be continuous across the surface. This condition is not satisfied by the ideal flow, so that a thin boundary layer forms at the bubble surface. As velocity perturbations are everywhere small in the layer, the governing equations

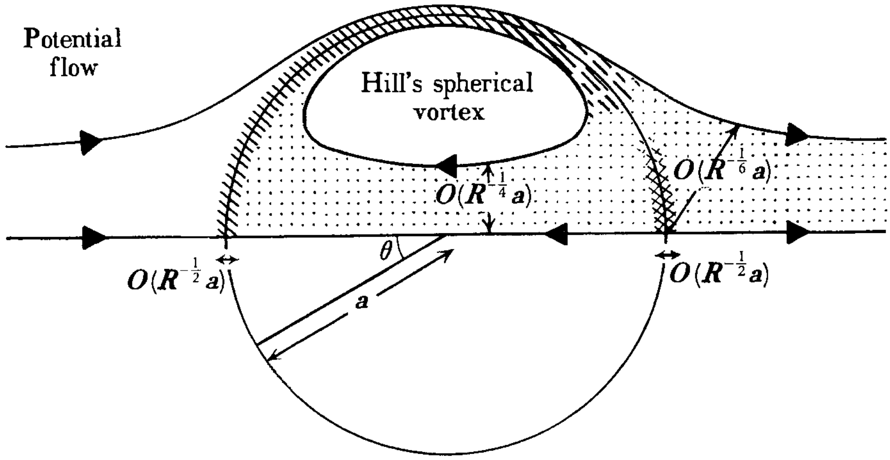

FIGURE 1. The form of the boundary layers and wakes. Diagonally shaded: the stress-induced viscous boundary layer (§3). Dotted: the inviscid stagnation regions and wakes (§3). Cross-hatched: the inner viscous boundary layer near the rear stagnation point (§6).

are linear, to a first approximation. Moore (1963)† discussed the detailed structure of this boundary layer and showed that it separated at the rear stagnation point of the bubble to form a thin wake.

Several authors (Conkie & Savic 1953; Garner & Haycock 1959; Chao 1962; Winnikow & Chao 1966) have pointed out that the analogous ideal flow for a drop is still potential flow outside it, with Hill's (1894) spherical vortex inside. They recognized that there would have to be a boundary layer at the drop's surface, but did not discuss the separation of the layer at the rear stagnation point in sufficient detail. Clearly, fluid discharged there from the interior boundary layer will move *forwards* in an interior 'wake' to re-enter the boundary layer at the front stagnation point (see figure 1). An adequate discussion of the boundary layer cannot be given unless the structure of the internal wake is taken into account. Winnikow & Chao's analysis, which assumed that no vorticity entered either the exterior or the interior boundary layer at the front stagnation point, is thus incorrect, except for the special case of a gas bubble where the internal flow has negligible effect on the external flow.

In §2 we discuss the spherical vortex and potential flow; by calculating their viscous dissipation we obtain a first approximation to the drag coefficient of the drop. In §3 we consider in detail the structure of the boundary layers and of the interior wake. The separation and reattachment of the interior boundary layer are discussed with the aid of analysis developed in DM for the gas bubble. Diffusion of vorticity is negligible in the interior and the separation and reattach-

† Henceforth referred to as DM.

ment regions, and the compatibility of the ejected and re-entrant vorticity distributions is shown to lead to an integral equation for the starting profile of the interior boundary layer. This integral equation is used in §4 to obtain a first approximation to the momentum defect in the exterior wake, and hence the drag. It confirms the result of §2; that it should is a necessary condition for the validity of the analysis, not satisfied by Winnikow & Chao's first approximation.

The method of numerical solution of our integral equation is given in §5; analytic work will be described elsewhere. It leads to the following approximate expression for the strength of the Hill's spherical vortex (equation (5.15)):

$$\frac{\text{Actual strength}}{\text{Inviscid-theory strength}} = 1 - \frac{2\cdot 5}{R_0^{\frac{1}{2}}}\frac{\{2\cdot 0 + (\mu_0\rho_0/\mu_1\rho_1)^{\frac{1}{2}}\}(1+1\cdot 5\mu_1/\mu_0)}{\{1\cdot 5 + (\mu_0\rho_0/\mu_1\rho_1)^{\frac{1}{2}}\}},$$

where μ_0 and μ_1 are the viscosities and ρ_0, ρ_1 the densities of the outer and inner fluids respectively, and where R_0 is the Reynolds number $2a\rho_0 U/\mu_0$, U being the speed of the drop and a the radius.

In order to find a better approximation to the drag than that of §2, the flow near the stagnation points must be examined more closely than in §3. The flow pattern is described in §6 and a momentum-integral approximation for the boundary layers near the stagnation points is found in §7. The results of §§5, 6, 7 are used in §8 to find the drag coefficient up to terms $O(R_0^{-\frac{3}{2}})$, thereby improving on the result of §2, which gave only the term $O(R_0^{-1})$.

2. The first approximation to the flow

We use spherical polar co-ordinates (r, θ, ϕ) whose centre O is at the centre of the drop and whose axis Oz is antiparallel to the undisturbed uniform stream with velocity U at infinity. Then the flow outside the drop, whose surface is given to a first approximation by the sphere $r = a$, is the potential flow

$$\bar{\psi}_0 = -\tfrac{1}{2}Ur^2\sin^2\theta(1-a^3/r^3), \tag{2.1}$$

where $\bar{\psi}_0$ is the Stokes streamfunction. We subscript flow quantities outside the drop with a '$_0$' and inside with a '$_1$'; subscript 'i' takes the values 0, 1. Otherwise the notation is the same as that of DM, so that $(\bar{q}_{ir}, \bar{q}_{i\theta}, 0)$ are the spherical polar velocity components, an overbar denotes quantities corresponding to the first approximation, and its absence those of the second approximation.

Inside the drop the fluid particles move in closed paths, so that the vorticity must be completely diffused (Batchelor 1956). This condition and the condition that $r = a$ is a stream-surface is satisfied uniquely by Hill's spherical vortex, for which

$$\bar{\psi}_1 = \tfrac{1}{2}Ar^2\sin^2\theta(a^2-r^2). \tag{2.2}$$

As pointed out by Batchelor (1956), this happens to be an exact solution of the full Navier–Stokes equations, and so is $\bar{\psi}_0$, being the streamfunction of a potential flow. These solutions therefore satisfy both the equations of motion and the condition that $r = a$ is a streamline. In addition, the tangential velocity components $\bar{q}_{i\theta}$ are taken to be continuous at $r = a$. As is well known, this requires that

$$A = 3U/(2a^2) \tag{2.3}$$

(see Lamb 1932, p. 246). The tangential stress components $\overline{p}_{ir\theta}$ at $r = a$ are now determined, and we find that

$$\overline{p}_{0r\theta} = -3\mu_0 U a^{-1} \sin\theta, \tag{2.4}$$

$$\overline{p}_{1r\theta} = \tfrac{9}{2}\mu_1 U a^{-1} \sin\theta, \tag{2.5}$$

where μ_i are the dynamic viscosities outside and inside. Obviously there must always be a discontinuity of shear stress at the surface of the drop, and the flow field $\overline{\psi}_i$ defined by (2.1), (2.2) and (2.3) cannot be an exact solution of the problem.

The choice of the rate of internal circulation implied by (2.3) is, strictly speaking, not yet justified. However, we remark that it makes the stresses near the bubble surface $O(\mu_i U a^{-1})$. If we did not insist on continuity of the tangential velocity across $r = a$, the discontinuity would give rise to a viscous smoothing layer, in which the tangential stress would be $O(\mu_i U a^{-1} R_i^{\frac{1}{2}})$. It seems reasonable to suppose that the internal motion will adjust itself to remove this larger stress.

There will also be a discontinuity of normal stress across the surface, arising primarily from the different hydrostatic pressures inside and out, whose effect will be to distort the drop from spherical shape. Our assumption of large surface tension enables us to neglect the distortion (but severely limits the range of experiments to which the theory can be applied).

In order to assess the validity of our approximation that the solution is nearly $\overline{\psi}_i$, let us think of the uniform stream as being subject to a surface stress $-F_{r\theta}$, acting in the surface $r = a$, where $-F_{r\theta}$ is just such as to annul the stress discontinuity. Thus

$$F_{r\theta} = 3Ua^{-1}(\mu_0 + \tfrac{3}{2}\mu_1)\sin\theta, \tag{2.6}$$

and $\overline{\psi}_i$ is an exact solution of this modified problem, and we can ask how it would respond if the imposed stress distribution $-F_{r\theta}$ were to be 'switched off'.† A more detailed discussion will be found in §3, but we can see that the change induced in the first approximation should be small, so long as both the external and internal Reynolds numbers, R_0 and R_1, defined by

$$R_i = 2aU\rho_i/\mu_i \tag{2.7}$$

are large. This is because the stress $F_{r\theta}$ is then small compared to the inertia stresses whose order of magnitude is $\rho_i U^2$.

We can use $F_{r\theta}$ to help us find the viscous dissipation in the motion given by $\overline{\psi}_i$ and thus obtain a first approximation to the drag on the drop. When $-F_{r\theta}$ acts on the fluid, its rate of working plus the rate of working of the pressure forces at infinity which are driving the flow must be equal, in a steady state, to the net viscous dissipation. But, since $\overline{\psi}_0$ is symmetric about the plane $\theta = \frac{1}{2}\pi$, the forces at infinity can do no net work, and the total rate of dissipation, Φ, is given by

$$\begin{aligned} \Phi &= \int_0^\pi F_{r\theta}(\overline{q}_{i\theta})_{r=a} 2\pi a^2 \sin\theta\, d\theta \\ &= 12U^2\pi a(\mu_0 + \tfrac{3}{2}\mu_1). \end{aligned} \tag{2.8}$$

† The idea of representing the difference between an approximate and an exact solution as a force acting on the former is due to Lamb (1932, p. 609).

Thus if we define the drag coefficient C_D by

$$\tfrac{1}{2}\rho_0 U^2 \pi a^2 C_D = \text{drag force} = \bar{\Phi}/U,$$

we obtain
$$C_D = \frac{48}{R_0}\left(1+\frac{3\mu_1}{2\mu_0}\right). \tag{2.9}$$

Levich (1949) obtained the special case $\mu_1 = 0$ of this result, corresponding physically to a vacuous (or, to a good approximation, gas) bubble.

3. The stress-induced boundary layer

If at some instant the imposed stress $-F_{r\theta}$ of the previous section is turned off, the surface of the drop will begin to act as a source of vorticity which will immediately start to diffuse away. Vorticity which diffuses outwards will be convected towards the back of the drop and then carried off downstream. Vorticity which diffuses inwards will, however, remain trapped inside the drop, and will eventually diffuse to every interior point. Thus the interior vorticity will be modified everywhere from the value implied by $\bar{\psi}_1$.

Since the Reynolds numbers are large, vorticity gradients† can exist in the final steady state, only (i) in a thin layer around the drop's surface, whose thickness is $O(aR_i^{-\frac{1}{2}})$ for the same reasons as in DM, and (ii) in external and internal wakes along the axis of symmetry. Outside the drop and its boundary layer and wake there will still be potential flow; inside it (and away from the internal boundary layer and wake), Batchelor's (1956) arguments will apply, and the flow will be Hill's spherical vortex, but with strength slightly different from its previous value because of the vorticity diffused inwards during the transient motion.

Let us write
$$\mathbf{q}_i' = \bar{\mathbf{q}}_i + \mathbf{q}_i,$$

where $\mathbf{q}_i'$ is the actual velocity field and $\bar{\mathbf{q}}_i$ that of the first approximation $\bar{\psi}_i$. The role of the remaining term, $\mathbf{q}_i$, is evidently to annul the stress discontinuity $O(\mu_i U/a)$ which the velocity field $\bar{\mathbf{q}}_i$ has at the drop surface.

As vorticity gradients must be confined to the thin boundary layer of thickness $O(aR_i^{-\frac{1}{2}})$,
$$\mu_i|\mathbf{q}_i|/(aR_i^{-\frac{1}{2}}) = O(\mu_i U/a),$$

giving
$$|\mathbf{q}_i| = O(UR_i^{-\frac{1}{2}}), \tag{3.1}$$

so that the perturbations to the basic flow are indeed small.

The boundary-layer equations for the perturbations can be derived in exactly the same way as for the spherical bubble. On writing, for $i = 0$ or 1,

$$\left.\begin{aligned} \delta_i^2 &= \mu_i/(\rho_i a U) = 2/R_i, \\ q_{i\theta} &= U\delta_i u_i, \\ r-a &= a\delta_i y, \end{aligned}\right\} \tag{3.2}$$

where $q_{i\theta}$ is the θ-component of the perturbation velocity $\mathbf{q}_i$, and δ_i, u_i and y are non-dimensional, it can be shown (see DM) that u_i satisfies the equation

$$\frac{3}{2}\frac{\partial}{\partial\theta}(u_i\sin\theta) - 3y\cos\theta\,\frac{\partial u_i}{\partial y} = \frac{3}{2}\frac{\partial}{\partial\theta}(u_i^\infty\sin\theta) + \frac{\partial^2 u_i}{\partial y^2}, \tag{3.3}$$

† Strictly, gradients of (vorticity)/$r\sin\theta$.

where u_i^∞ are the asymptotic values of u_i at the inner ($i = 1$) and outer ($i = 0$) limits of the boundary layer.

Consideration of the exterior flow shows at once that $u_0^\infty = o(1)$. The interior flow is a perturbed Hill's vortex, so we have, in all,

$$\left.\begin{aligned} u_0^\infty &= 0, \\ u_1^\infty &= \tfrac{3}{2} C \sin\theta. \end{aligned}\right\} \tag{3.4}$$

We do not at this stage know the exact strength of the vortex, which is $(1 + C\delta_1)$ times its value for an ideal fluid, but we can say, in view of (3.1), that C is of order unity. It seems plausible physically to guess that the actual vortex will be slightly retarded, making C negative. This will be confirmed later when C is evaluated.

The velocity field must be continuous across $r = a$, which makes

$$\delta_0 \lim_{y\to 0+} u_0(y,\theta) = \delta_1 \lim_{y \to 0-} u_1(y,\theta), \tag{3.5}$$

while the condition of continuity of stress across $r = a$ leads to

$$\mu_0 \lim_{y\to 0+} \frac{\partial u_0}{\partial y} - \mu_1 \lim_{y\to 0-} \frac{\partial u_1}{\partial y} = 3\sin\theta(\mu_0 + \tfrac{3}{2}\mu_1). \tag{3.6}$$

Equation (3.3) is parabolic, so that, as well as (3.4), (3.5) and (3.6), a boundary condition on some line $\theta =$ constant is needed. To deduce it we must discuss the overall flow pattern (figure 1).

As we have seen, vorticity which diffuses inwards will be swept forward along the internal wake and will re-enter the interior boundary layer near the front stagnation point of the drop. The wake is effectively inviscid, on account of its breadth which is $O(aR^{-\frac{1}{4}})$, or $O(R^{+\frac{1}{4}})$ times that of the boundary layer. (DM's argument establishing this result for the external wake of a bubble still holds here.) The vorticity which leaves the interior boundary layer at the rear stagnation point therefore re-enters that layer unchanged, to a first approximation, at the front stagnation point. This is our required condition, together with the obvious one that no vorticity enters the boundary layer from outside the drop at the front stagnation point.

To apply it, however, we must discuss the way in which the boundary layer separates from the surface of the drop at the rear and reattaches at the front stagnation point. The dynamics of the separation was examined in DM for the special case of a vacuous bubble, and it was shown that in the separation region, which extends to a distance $O(aR^{-\frac{1}{8}})$ from the stagnation point, the flow was inviscid. The reason is that the large curvature of the streamlines there introduces large inertial forces, while the viscous forces, which depend mainly on the thickness of the region, are not correspondingly increased. Moreover, the velocity field is still only slightly disturbed from the first approximation in the separation region if the Reynolds number is sufficiently large. This fortunate circumstance (which does not obtain in the corresponding two-dimensional flow: see Harper (1963)) is due to the fact that the vorticity is weakened by the contraction of the vortex-lines as they approach the axis of symmetry.

As the internal wake is also inviscid, we can assert that in the stagnation regions of the drop and the internal wake joining them, vorticity is convected with negligible diffusion along the streamlines of the first approximation to the flow. This implies that in the stagnation-wake region the vorticity ω satisfies the equation

$$\frac{\omega}{r\sin\theta} = B_i(\overline{\Psi}_i). \tag{3.7}$$

It was shown in DM how $B_i(\overline{\Psi}_i)$ could be determined by matching to the boundary-layer solution.

We thus attack the problem in the following way. First we solve the system (3.3)–(3.6), imposing at $\theta = 0$ zero 'input' of vorticity for $y > 0$ and arbitrary input for $y < 0$. That determines the output distribution at $\theta \to \pi$, and (3.7) then requires that the input and output vorticity distributions be the same. This imposes a condition on the input distribution, which will appear in the form of an integral equation.

Let us now turn to the details. It helps to transform (3.3) into the diffusion equation, which we do by the method of Chao (1962). With the substitutions

$$Y = y\sin^2\theta, \tag{3.8}$$

$$X = \tfrac{2}{9}(2-3\cos\theta+\cos^3\theta), \tag{3.9}$$

$$f_i(X, Y) = \sin\theta(u-u_i^\infty), \tag{3.10}$$

equation (3.3) becomes

$$\partial f_i/\partial X = \partial^2 f_i/\partial Y^2. \tag{3.11}$$

The continuity condition for velocity (3.5) becomes

$$\delta_0 \lim_{Y\to 0+} f_0(X, Y) = \delta_1 \lim_{Y\to 0-} f_1(X, Y) + \tfrac{3}{2}C\delta_1 \sin^2\theta(X), \tag{3.12}$$

where $\theta(X)$ is found by solving the cubic equation (3.9) for $\cos\theta$, which gives a unique θ in $0 \leqslant \theta \leqslant \pi$ for each X in $0 \leqslant X \leqslant \frac{8}{9}$. The continuity condition for stress (3.6) gives

$$\mu_0 \lim_{Y\to 0+} \frac{\partial f_0}{\partial Y} = \mu_1 \lim_{Y\to 0-} \frac{\partial f_1}{\partial Y} + 3(\mu_0 + \tfrac{3}{2}\mu_1). \tag{3.13}$$

The definition of u_1 and (3.10) obviously require that

$$\lim_{Y\to\infty} f_0(X, Y) = \lim_{Y\to -\infty} f_1(X, Y) = 0. \tag{3.14}$$

Finally, because

$$\overline{\Psi}_i \sim \tfrac{3}{2}U\delta_i a^2 Y \tag{3.15}$$

near both $\theta = 0$ and $\theta = \pi$, it can be shown, by symmetry of the inviscid streamfunction $\overline{\Psi}_i$ about the equatorial plane of the drop, and from equation (3.7) which implies that perturbation vorticity is carried up the inner wake from the rear to the front stagnation region without change, that

$$f_1(0, Y) = f_1(X_e, Y) = \gamma(Y), \text{ say, for } Y < 0, \tag{3.16†}$$

where $X_e = \frac{8}{9}$, the value of X in (3.9) at $\theta = \pi$, and where we have, with consequences to be examined in detail in §§6 and 7, related the distributions at $X = 0$ and $X = X_e$ rather than those at $R^{-\frac{2}{3}} \ll X \ll 1$ and $R^{-\frac{2}{3}} \ll X_e - X \ll 1$, as would be strictly required (see figure 1). From (3.9),

$$X \sim \tfrac{1}{6}\theta^4 \text{ near } \theta = 0,\; X_e - X \sim \tfrac{1}{6}(\pi-\theta)^4 \text{ near } \theta = \pi.$$

† This consistency condition also ensures that the pressure is single-valued.

The differential equation (3.11) must now be solved with boundary conditions (3.12) to (3.14) and initial conditions

$$f_0(0, Y) = 0 \quad \text{on} \quad Y > 0, \tag{3.17}$$

$$f_1(0, Y) = \gamma(Y) \quad \text{on} \quad Y < 0, \tag{3.18}$$

where $\gamma(Y)$ is still to be determined.

The solution is straightforward but lengthy, and the details are deferred to the appendix. We obtain

$$\begin{aligned} f_0(X, Y) = {} & \{-\delta_1(6\mu_0+9\mu_1)X^{\frac{1}{2}}\phi(Y/2X^{\frac{1}{2}})+\tfrac{3}{2}\mu_1\delta_1 CN(X, Y)\}/(\delta_0\mu_1+\delta_1\mu_0) \\ & +\frac{\mu_1\delta_1}{\delta_0\mu_1+\delta_1\mu_0}\frac{1}{(\pi X)^{\frac{1}{2}}}\int_{-\infty}^{0}\gamma(Y')\exp\{-(Y-Y')^2/4X\}dY', \end{aligned} \tag{3.19}$$

$$\begin{aligned} f_1(X, Y) = {} & \{-\delta_0(6\mu_0+9\mu_1)X^{\frac{1}{2}}\phi(-Y/2X^{\frac{1}{2}})-\tfrac{3}{2}\mu_0\delta_1 CN(X, -Y)\}/(\delta_0\mu_1+\delta_1\mu_0) \\ & +\frac{1}{2(\pi X)^{\frac{1}{2}}}\int_{-\infty}^{0}\gamma(Y')[\exp\{-(Y-Y')^2/4X\} \\ & +\frac{\delta_0\mu_1-\delta_1\mu_0}{\delta_0\mu_1+\delta_1\mu_0}\exp\{-(Y+Y')^2/4X\}]dY', \end{aligned} \tag{3.20}$$

where
$$\phi(t) = \pi^{-\frac{1}{2}}e^{-t^2}-t\operatorname{erfc} t = i\operatorname{erfc} t \tag{3.21}$$

in the notation of Carslaw & Jaeger (1959), and

$$N(X, Y) = \frac{Y}{2\pi^{\frac{1}{2}}}\int_0^X \frac{\sin^2\theta(\lambda)}{(X-\lambda)^{\frac{3}{2}}}\exp\left\{\frac{-Y^2}{4(X-\lambda)}\right\}d\lambda. \tag{3.22}$$

The consistency condition (3.16) then leads to the following integral equation which must be satisfied by $\gamma(Y)$:

$$\begin{aligned} & \frac{1}{2(\pi X_e)^{\frac{1}{2}}}\int_{-\infty}^{0}\gamma(Y')\left[\exp\left\{-\frac{(Y-Y')^2}{4X_e}\right\}+\frac{\delta_0\mu_1-\delta_1\mu_0}{\delta_0\mu_1+\delta_1\mu_0}\exp\left\{-\frac{(Y+Y')^2}{4X_e}\right\}\right]dY' \\ & = \gamma(Y)+\frac{1}{\delta_0\mu_1+\delta_1\mu_0}\left\{X_e^{\frac{1}{2}}\delta_0(6\mu_0+9\mu_1)\phi\left(-\frac{Y}{2X_e^{\frac{1}{2}}}\right)+\tfrac{3}{2}\mu_0\delta_1 CN(X_e, -Y)\right\}. \end{aligned} \tag{3.23}$$

We observe that, at this stage, the constant C which gives the strength of the spherical vortex is still unknown. But we have not yet used (3.14) in conjunction with (3.16), which makes $\gamma(Y)$ satisfy

$$\lim_{Y\to-\infty}\gamma(Y) = 0. \tag{3.24}$$

It will be seen in §5 that this constraint on the solution of the integral equation determines C uniquely.

4. The drag calculated from the wake structure

The functions $B_i(\psi_i)$ defined in (3.7) determine the structure of the stagnation regions and of the internal and external wakes. If the external wake is known, it should be possible to calculate a first approximation to the drag by considering the momentum defect in the wake. The result should agree with that given by the dissipation argument of §2, so that, in effect, one has a test of the overall dynamical consistency of the proposed flow. First, we obtain the function $B_0(\overline{\psi}_0)$.

As the vorticity in the exterior boundary layer is, to a first approximation, $\partial q_{0\theta}/\partial r$, we find from (3.7) and (3.15) that

$$B_0(\tfrac{3}{2}U\delta_0 a^2 Y) = \frac{U}{a^2} f_{0Y}(X_e, Y). \tag{4.1}$$

It can easily be shown (Moore 1965) that the drag force D can be expressed as

$$D = \frac{2\pi\rho_0}{U}\int_0^\infty \overline{\psi}_0 B_0(\overline{\psi}_0)\,d\overline{\psi}_0, \tag{4.2}$$

and on inserting (4.1) and partially integrating, using the fact that $f_0(X,\infty) = 0$, we have

$$D = -\frac{9\pi U a\mu_0}{2}\int_0^\infty f_0(X_e, Y)\,dY. \tag{4.3}$$

If we substitute for f_0 and carry out the integration we have

$$-\frac{2D}{9\pi U a\mu_0} = -\frac{\delta_1 X_e(6\mu_0+9\mu_1)}{2(\delta_0\mu_1+\mu_0\delta_1)} + \frac{3\mu_1\delta_1 C}{2(\delta_0\mu_1+\delta_1\mu_0)}\int_0^\infty N(X_e, Y)\,dY$$
$$+\frac{\mu_1\delta_1}{(\delta_0\mu_1+\delta_1\mu_0)}\int_{-\infty}^0 \gamma(Y')\,\mathrm{erfc}\left(-\frac{Y'}{2X_e^{\frac{1}{2}}}\right)dY'. \tag{4.4}$$

On integrating the integral equation (3.23) from $-\infty$ to 0 we find that

$$\int_{-\infty}^0 \gamma(Y')\,\mathrm{erfc}\left(-\frac{Y'}{2X_e^{\frac{1}{2}}}\right)dY' = -\frac{X_e\delta_0(6\mu_0+9\mu_1)}{2\mu_0\delta_1} - \tfrac{3}{2}C\int_0^\infty N(X_e, Y)\,dY, \tag{4.5}$$

and on substituting into (4.4) we recover the expression for C_D given in (2.9). The terms in C cancel out, as they must; we have found only a first approximation to the drag, and altering C by a finite amount can alter the drag only in higher approximations.

5. The solution of the integral equation

If we introduce a new variable z given by

$$z = -\tfrac{1}{2}(X_e)^{-\frac{1}{2}}Y = -\frac{3}{4\sqrt{2}}Y, \tag{5.1}$$

and define

$$\gamma(Y) = g(z), \tag{5.2}$$

the integral equation (3.23) becomes

$$\pi^{-\frac{1}{2}}\int_0^\infty g(z')\{e^{-(z-z')^2} + \lambda_a e^{-(z+z')^2}\}\,dz' - g(z) = \lambda_b\phi_b(z) + C\phi_c(z) = \chi(z), \text{ say}. \tag{5.3}$$

The constants λ_a, λ_b are defined in terms of the dimensionless ratios V, V' of fluid properties given by

$$V = \mu_0/\mu_1, \quad V' = (\mu_0\rho_0/\mu_1\rho_1)^{\frac{1}{2}} = \mu_0\,\delta_1/\mu_1\,\delta_0, \tag{5.4}$$

as

$$\lambda_a = \frac{1-V'}{1+V'}, \quad \lambda_b = \frac{2V+3}{2V'+3}, \tag{5.5}$$

and

$$\phi_b(z) = \frac{2\sqrt{2}(2V'+3)}{V'+1}\phi(z), \tag{5.6}$$

$$\phi_c(z) = \frac{3V'}{2(V'+1)} N(X_e, -2X_e^{-\frac{1}{2}}z)$$

$$= \frac{9V'z}{(V'+1)\sqrt{\pi}} \int_{-1}^{1} \frac{(1-\mu^2)^2}{(2+3\mu-\mu^3)^{\frac{3}{2}}} \exp\left\{\frac{-4z^2}{2+3\mu-\mu^3}\right\} d\mu. \tag{5.7}$$

Our procedure for solving (5.3) was to write

$$g(z) = \lambda_b g_b(z) + C g_c(z), \tag{5.8}$$

so that $g_b(z)$ must satisfy

$$\pi^{-\frac{1}{2}} \int_0^\infty g_b(z')\{e^{-(z-z')^2} + \lambda_a e^{-(z+z')^2}\} dz' - g_b(z) = \phi_b(z), \tag{5.9}$$

and $g_c(z)$ a similar equation with b replaced by c. The solutions are rendered unique by requiring that $g_b(z)$ and $g_c(z)$ have finite limits as $z \to \infty$; it is not difficult to show that eigen-solutions exist, but none which are finite at infinity if $V' > 0$. When $g_b(z)$ and $g_c(z)$ have been found, we obtain C/λ_b from the requirement (3.25) that $g(\infty) = 0$, which gives

$$\frac{C}{\lambda_b} = -\frac{g_b(\infty)}{g_c(\infty)}. \tag{5.10}$$

The integral equations were solved numerically for $\lambda_a = \frac{2}{3}, \frac{1}{3}, 0, -\frac{1}{3}, -\frac{2}{3}, -1$, i.e. $V' = \frac{1}{5}, \frac{1}{2}, 1, 2, 5, \infty$. Before describing the method, we observe that, if $\lambda_a = 0$, i.e. $\mu_0\rho_0 = \mu_1\rho_1$, the equation is greatly simplified, since the second exponential term in its kernel vanishes. The resulting integral equation can then be attacked by the Wiener–Hopf method, though on account of the complexity of $\phi_c(z)$ a solution in closed form is impracticable. What can be done is to solve the homogeneous equation

$$\pi^{-\frac{1}{2}} \int_0^\infty e^{-(z-z')^2} h(z') dz' = h(z), \tag{5.11}$$

and to determine C/λ_b from the easily established result

$$\int_0^\infty h(z)\{\lambda_b \phi_b(z) + C\phi_c(z)\} dz = 0. \tag{5.12}$$

However, the details of the solution of (5.11) are rather lengthy, and will be described elsewhere. The value of C/λ_b obtained from (5.12) gave a useful check on our numerical procedure.

Equation (5.9) and its analogue for $g_c(z)$ were solved iteratively. Using values of z equally spaced at interval Δ, we put an assumed set $g_0(z)$ of values of $g(z)$ in the integral, which was evaluated by Simpson's rule, to find a new set, $g_1(z)$. A new function, $g_2(z) = g_1(z) + \text{constant}$, was then obtained from $g_1(z)$ by requiring that

$$\int_0^\infty g_2(z) \operatorname{erfc} z\, dz = -\frac{1+V'}{V'} \int_0^\infty \chi(z) dz, \tag{5.13}$$

which is found by integrating (5.3) from 0 to ∞. Then $g_0(z)$ was replaced by

$$g_0(z) \to g_2(z) + k\{g_2(z) - g_0(z)\}, \tag{5.14}$$

and the cycle repeated. In (5.14) k is a constant chosen to make the iteration converge as fast as possible; the best value found was 0·78. Forty iterations usually sufficed to give the values of $g(z)$ correct to four significant digits. To that accuracy, each $g_n(z)$ could be assumed constant for $z > 5$, and when $\chi(z) = \phi_b(z)$ the steplength Δ could be taken as 0·2. Because $\phi_c(z)$ exists but has no derivative at the origin, the error in the numerical integrations is then of order Δ^2 rather

v'...	0·2	0·5	1·0	2·0	5·0	∞
$3\lambda_a$...	2	1	0	−1	−2	−3
z			$g_b(z)$			
0	−22·18	−10·868	−7·071	−5·153	−3·985	−3.192
0·2	−21·53	−10·239	−6·470	−4.581	−3.444	−2.682
0·4	−21·14	−9·865	−6.112	−4·240	−3.119	−2·376
0·6	−20·94	−9·669	−5·924	−4·060	−2·949	−2·214
0·8	−20·85	−9·585	−5·843	−3·982	−2·874	−2·143
1·0	−20·82	−9·558	−5·817	−3·957	−2·850	−2·120
1·2	−20·82	−9·556	−5·815	−3·955	−2·848	−2·118
1·4	−20·82	−9·561	−5·819	−3·959	−2·852	−2·122
1·6	−20·83	−9·565	−5·823	−3·963	−2·855	−2·125
1·8	—	−9·567	−5·825	−3·965	−2·857	−2·127
2·0	—	—	−5·826	−3·966	−2·858	—
2·2	—	—	—	—	−2·857	—
∞	−20·83	−9·567	−5·826	−3·966	−2·857	−2·127

TABLE 1. Solutions of the integral equation (5.9) for $g_b(z)$

v'...	0·2	0·5	1·0	2·0	5·0	∞
$3\lambda_a$...	2	1	0	−1	−2	−3
z			$g_c(z)$			
0	−0·8614	−0·6950	−0·5258	−0·3537	−0·1785	0
0·2	−1·027	−1·027	−1·026	−1·025	−1·023	−1·019
0·4	−1·054	−1·084	−1·114	−1·144	−1·175	−1·207
0·6	−1·057	−1·090	−1·124	−1·159	−1·196	−1·235
0·8	−1·054	−1·084	−1·116	−1·150	−1·186	−1·223
1·0	−1·051	−1·079	−1·108	−1·140	−1·174	−1·210
1·2	−1·050	−1·076	−1·104	−1·134	−1·167	−1·202
1·4	−1·049	−1·075	−1·103	−1·132	−1·164	−1·199
1·6	—	—	−1·102	−1·132	−1·164	−1·198
1·8	—	—	−1·103	−1·133	−1·165	−1·199
2·0	—	—	—	—	—	−1·199
2·2	—	—	—	—	—	−1·200
∞	−1·049	−1·075	−1·103	−1·133	−1·165	−1·200
$\dfrac{g_b(\infty)}{g_c(\infty)}$	19·86	8·900	5·282	3·500	2·452	1·772
$\dfrac{2{\cdot}5(2+V')}{\sqrt{2V'}}$	19·45	8·839	5·303	3·536	2·475	1·768

TABLE 2. Solutions of the integral equation (5.9) for $g_c(z)$

than the usual Δ^4 for Simpson's rule, and for $\chi(z) = \phi_c(z)$ the calculations had to be performed for $\Delta = 0{\cdot}2$ and $\Delta = 0{\cdot}1$, and the results extrapolated to $\Delta = 0$. (A few values of $g_c(z)$ were also found with $\Delta = 0{\cdot}05$ to check on the extrapolation.)

The results are given in tables 1 and 2, together with the values of $g_b(\infty)/g_c(\infty)$ for the stated range of values of V'. As $2{\cdot}5(V'+2)/V'\sqrt{2}$ turns out to be a fairly good simple approximation to $g_b(\infty)/g_c(\infty)$, equation (5.10) implies that the internal circulation is less than that of Hill's classical theory by a factor approximately equal to

$$1+C\delta_1 = 1-\frac{2{\cdot}5}{R_1^{\frac{1}{2}}}\frac{V'+2}{V'}\frac{2V+3}{2V'+3} = 1-\frac{2{\cdot}5}{R_0^{\frac{1}{2}}}\frac{V'+2}{V}\frac{2V+3}{2V'+3}, \tag{5.15}$$

provided that this factor is nearly unity. If it is not, the Reynolds number is obviously not high enough for the perturbations to be really small, and higher-order terms are likely to be important. That is likely to happen if V is small, so that our theory is not applicable to drops falling in a gas or in a much less viscous liquid.

It is of some interest that both $g_b(z)$ and $g_c(z)$ oscillate towards their asymptotic values. This appears to be a general property of boundary-layer solutions around regions of closed streamlines, having been found in completely different contexts by Mills (1965) and Burggraf (1966). The latter explained it by considering the propagation of 'waves' of vorticity away from the bounding surface through the boundary layer.

6. The nature of the flow near the stagnation points

In this section, it is shown that the flow field near the stagnation points has a complicated structure of inner viscous boundary layers, determined by the main boundary layers calculated in §§3 and 5. (Near the front stagnation point, the inner layer is really the initial portion of the main layer.) The viscous dissipation in these inner layers does not contribute to (2.9), but it must be included when seeking higher approximations to the drag, except for a gas bubble whose internal density and viscosity are negligibly small. In fact, on this account the correction to (2.9) is $O(R^{-\frac{2}{3}}\ln R)$, in contrast to the $O(R^{-\frac{1}{2}})$ correction to Levich's (1949) result for gas bubbles, found in DM. (For the purpose of these order-of-magnitude considerations, we do not need to distinguish between R_0 and R_1.) However, in the range of Reynolds numbers of interest, most of the dissipation actually comes from the boundary layer and external wake already described. Thus a precise treatment of the inner viscous boundary layers is not needed. In this section we describe their structure in general terms and in §7 we give a momentum-integral treatment of them which is sufficiently accurate for the purpose of estimating their dissipation and showing that it is negligible.

Let us start with the rear stagnation region. It was shown in §3 that when $\pi-\theta = O(R^{-\frac{1}{6}})$ the boundary-layer equations would break down, to be replaced by an inviscid equation which prescribes the perturbation vorticity as a function of the zeroth-order streamfunction $\overline{\Psi}$ (see equation (3.7)). This conclusion must apply to both the exterior and interior rear stagnation regions of the drop. The perturbation velocity field is determined from the perturbation vorticity field by requiring that it have zero normal component at the drop surface, zero radial component on the axis of symmetry, and that it tend to zero outside the stagnation region. Clearly there is no reason to expect that the tangential components of perturbation velocity on the two sides of the drop surface, $q_{0\theta}^{(0)}$ and

$q_{1\theta}^{(0)}$ say, will be the same. If they differ, there must be an inner viscous boundary layer between the two stagnation regions (figure $2a$).

In the case of the gas bubble (see DM), the interior motion can be consistently ignored, when calculating the drag and distortion. Consequently, so can the difference between $q_{0\theta}^{(0)}$ and $q_{1\theta}^{(0)}$, and the inner viscous boundary layer at the rear stagnation region need not be considered.

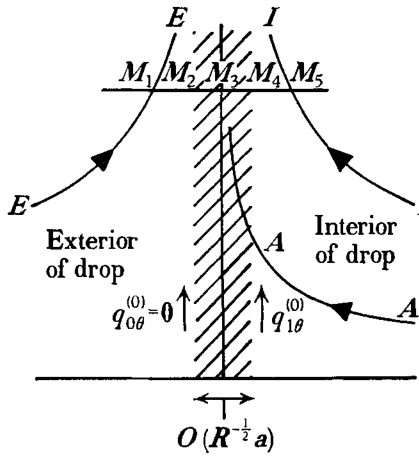

FIGURE $2a$

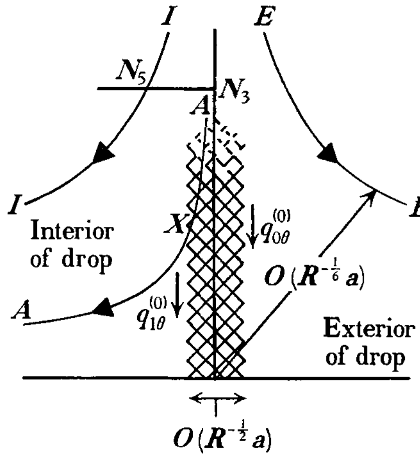

FIGURE $2b$

FIGURES $2a$, b. Details of the flow patterns near the stagnation points (front, figure $2a$; rear, figure $2b$). Viscous boundary layers are shaded as in figure 1.

The boundary-layer solutions (3.19) and (3.20) can be shown to hold for $\pi-\theta \gg R^{-\frac{1}{6}}$, by using the estimates for the terms neglected in the full equations given in DM, so that any discontinuity in tangential velocity will be negligible until $\pi-\theta = O(R^{-\frac{1}{6}})$. Thus the effective length of the inner viscous boundary layer is $O(aR^{-\frac{1}{6}})$, while its thickness can be shown to be $O(aR^{-\frac{1}{2}})$, as usual. The tangential components of perturbation velocity are $O(UR^{-\frac{1}{3}})$ in the stagnation regions, and the viscous dissipation rate in this inner viscous boundary layer is then readily estimated to be $O(\rho U^3a^2R^{-\frac{3}{2}})$.

Figure $2b$ shows the details of the rear stagnation region. II and EE are streamlines of the basic flow, for which $\overline{\psi} = O(Ua^2R^{-\frac{1}{2}})$. They enclose the region containing perturbation vorticity convected out of the boundary layer on the drop surface, and their equations are of the form

$$(r-a)(\pi-\theta)^2 = O(aR^{-\frac{1}{2}}). \tag{6.1}$$

This leads at once to the characteristic linear size $aR^{-\frac{1}{6}}$ for the stagnation regions as shown in DM, §3. The shaded region is the inner viscous boundary layer, and it starts gradually as the difference $q_{0\theta}^{(0)}-q_{1\theta}^{(0)}$ grows from its value $o(UR^{-\frac{1}{3}})$ for $\pi-\theta \gg R^{-\frac{1}{6}}$. AA is a typical streamline of the basic flow which passes through the inner viscous boundary layer to emerge at X. Clearly X has co-ordinates $\pi-\theta = O(R^{-\frac{1}{6}})$ and $r-a = O(aR^{-\frac{1}{2}})$, making $\overline{\psi} = O(Ua^2R^{-\frac{5}{6}}) \ll Ua^2R^{-\frac{1}{2}}$ on AA. This fact is important, since it implies that for large R a negligible proportion of the streamlines which enter the wakes have passed through the inner viscous boundary layer. Thus the matching procedure used to determine the wake structure, which assumes no diffusion of vorticity across these streamlines, remains valid.

At the forward stagnation point the situation is similar. In this case, however, there is no perturbation in the exterior stagnation region, so that $q_{0\theta}^{(0)} = 0$. But $q_{1\theta}^{(0)}$ is just minus its value at the corresponding point of the interior rear stagnation region, since our consistency condition (3.7) on the interior perturbation vorticity implies that the two interior stagnation regions are identical, apart from obvious sign changes. Consequently $q_{1\theta}^{(0)} - q_{0\theta}^{(0)}$ remains of order $UR^{-\frac{1}{3}}$ for $\theta \gg R^{-\frac{1}{6}}$, and the dissipation associated with the forward inner viscous boundary layer is of order greater than $\rho U^3 a^2 R^{-\frac{5}{6}}$. We give its actual order presently.

Figure 2a shows details of the forward stagnation region. As we have seen, the forward inner viscous boundary layer extends beyond the stagnation region itself, where $\theta = O(R^{-\frac{1}{6}})$. It becomes eventually the boundary layer proper which was described in §3. If $M_1M_2M_3M_4M_5$ is a cross-section of the flow at some θ' in the range $R^{-\frac{1}{6}} \ll \theta' \ll 1$, $q_{1\theta}^{(0)}$ and $q_{0\theta}^{(0)}$ above are the tangential velocities at M_2 and M_4, and their difference is reflected in the discontinuity of the functions $f_i(0, Y)$ of (3.17) and (3.18). We see that (3.17) is really valid on M_1M_2 and (3.18) on M_4M_5, i.e. $Y \gg \theta'^2$, $-Y \gg \theta'^2$ respectively, but that in the limit, as $R \to \infty$, θ' may $\to 0$, and we can replace these conditions by $Y > 0$, $Y < 0$. Moreover, almost all streamlines crossing M_3M_5 have not passed through any region where significant viscous forces are acting after leaving the corresponding section N_3N_5 in the rear stagnation region (figure 2a). This is because $M_3M_4/M_4M_5 = O(R^{-\frac{1}{3}})$. Thus the consistency condition (3.16) on the vorticity is asymptotically correct in the limit $R \to \infty$.

Thus we have an *a posteriori* justification of the solution given in §§3–5. It remains to consider the order of magnitude of the viscous dissipation associated with the boundary layer near the front stagnation point.

On making the usual boundary-layer assumptions, we find that this is of order

$$R_i^{-\frac{1}{2}} \rho_i U a^4 \int_0^{\theta'} \left(\frac{\partial q_{i\theta}}{\partial r}\right)^2 \theta \, d\theta, \tag{6.2}$$

where, as before, $R_i^{-\frac{1}{6}} \ll \theta' \ll 1$ and where we have carried out in order of magnitude the integration across the boundary layer.

Now it follows from the theory of §§3 and 5 that

$$q_{1\theta}^{(0)} = O(UR_1^{-\frac{1}{2}}/\theta) \tag{6.3}$$

for $R_1^{-\frac{1}{6}} \ll \theta < \theta'$, and from the analysis of §7 that

$$q_{1\theta}^{(0)} = O(UR_1^{-\frac{1}{3}}), \tag{6.4}$$

uniformly in θ, for $\theta = O(R_1^{-\frac{1}{6}})$. If we estimate the integrand in the dissipation integral (6.2), using these orders of magnitude, and assuming that in the boundary layer $q_{i\theta} = O(q_{i\theta}^{(0)})$, we find that the dissipation is of order

$$R_1^{-\frac{5}{6}} \rho_i U^3 a^2 \int_{R_1^{-\frac{1}{6}}}^{\theta'} \frac{d\theta}{\theta}, \tag{6.5}$$

which is of order $\rho_i U^3 a^2 R^{-\frac{5}{6}} \ln R_1$. We will recover this result in §8 and get a precise value of the numerical coefficient.

7. The inner viscous boundary layers

In order to estimate the viscous dissipation associated with the inner viscous boundary layers at the stagnation points, we must determine their structure. We will use a momentum integral approach rather than attempt an analytic solution. Since the dissipation associated with the inner viscous boundary layers turns out to be much smaller than that of either the principal boundary layer or the wake, the error in the net dissipation introduced by this approximation is relatively unimportant.

The first step is to determine the exterior velocities $q_{0\theta}^{(0)}$ and $q_{1\theta}^{(0)}$ defined in § 6. We consider in detail only the neighbourhood of the front stagnation point: the changes in our treatment needed to get $q_{0\theta}^{(0)}$ and $q_{1\theta}^{(0)}$ at the rear stagnation point are minor and are given later.

Near the stagnation point, curvature of the drop's surface can be ignored, which allows us to use a local system of dimensionless cylindrical polar co-ordinates (s_1, z_1) where

$$a - r = a\delta_1^{\frac{1}{3}} z_1, \quad \theta = \delta_1^{\frac{1}{3}} s_1, \tag{7.1}$$

so that $s_1 = 0$ is the axis of the drop and $z_1 = 0$ its surface. The variables z_1 and s_1 are thus $O(1)$ in the stagnation region. We also define a local dimensionless stream function ψ_1 by the equations

$$q_{1z} = U\delta_1^{\frac{2}{3}} \frac{1}{s_1} \frac{\partial \psi_1}{\partial s_1} \doteqdot -q_r, \tag{7.2}$$

$$q_{1s} = -U\delta_1^{\frac{2}{3}} \frac{1}{s_1} \frac{\partial \psi_1}{\partial z_1} \doteqdot q_\theta, \tag{7.3}$$

following DM, equations (3.18) to (3.21), except that our z_1 is measured in the opposite direction to DM's z. An argument analogous to that of DM, § 3, with our equations (3.7), (3.15) and (5.2), immediately gives

$$\frac{\partial^2 \psi_1}{\partial s_1^2} - \frac{1}{s_1}\frac{\partial \psi_1}{\partial s_1} + \frac{\partial^2 \psi_1}{\partial z_1^2} = \frac{3s_1^2}{4\sqrt{2}} \frac{dg(z)}{dz} = s_1^2 b_1(s_1^2 z_1), \tag{7.4}$$

where

$$z = 3s_1^2 z_1 / 4\sqrt{2} \tag{7.5}$$

and

$$b_1(s_1^2 z_1) = \frac{a^2}{U} B_1(-\tfrac{3}{2} U a^2 \delta_1 s_1^2 z_1) = \frac{a^2}{U} B_1(\overline{\psi}_1). \tag{7.6}$$

The boundary conditions of (7.4) are that $s_1 = 0$ and $z_1 = 0$ be both streamlines, $\psi = 0$ for definiteness, and that both velocity components should tend to zero at the origin and infinity (see DM for the reasons). By an obvious use of hydrodynamic images, and with the help of Lamb (1932, Art. 161 equation (14)) giving the streamfunction of an isolated vortex ring, which we use as a Green's function, we find

$$\frac{\partial \psi_1}{\partial z_1}(s_1, 0) = \int_0^\infty \int_0^\infty \int_0^\infty s^2 s_1 k e^{-kz_1} J_1(ks_1) J_1(ks) b_1(s^2 z_1) dz_1 ds dk, \tag{7.7}$$

which gives q_{1s} (by equation (7.3)) on the surface, $z_1 = 0$, as a function of s_1.

Now it follows from the theory of §§3 and 5 that

$$q_{1\theta}^{(0)} \sim U\delta_1 g(0)/\theta \tag{7.8}$$

for

$$\theta \gg R_1^{-\frac{1}{4}}.$$

Hence, for consistency we require that $\partial\psi_1/\partial z_1 \to -g(0)$ as $s_1 \to \infty$. This is readily verified from (7.7), for, if we put

$$k = \kappa/s_1, \quad s = s_1\sigma, \quad z_1 = \zeta/s_1^2,$$

that equation becomes

$$\frac{\partial\psi_1}{\partial z_1}(s_1, 0) = \int_0^\infty\int_0^\infty\int_0^\infty \sigma^2\kappa e^{-\kappa\zeta/s_1^3} J_1(\kappa) J_1(\kappa\sigma)\, b_1(\sigma^2\zeta)\, d\zeta\, d\sigma\, d\kappa$$
$$\sim \int_0^\infty\int_0^\infty\int_0^\infty \kappa\sigma^2 J_1(\kappa) J_1(\kappa\sigma)\, b_1(\sigma^2\zeta)\, d\zeta\, d\sigma\, d\kappa$$

in the limit as $s_1 \to \infty$, and this last integral

$$= \int_0^\infty b_1(t)\, dt \int_0^\infty\int_0^\infty \kappa J_1(\kappa) J_1(\kappa\sigma)\, d\sigma\, d\kappa$$
$$= -g(0),$$

because

$$\int_0^\infty J_1(x)\, dx = 1 \quad \text{and} \quad b_1(t) = \frac{d}{dt} g\left(\frac{3t}{4\sqrt{2}}\right).$$

Near $s_1 = 0$, on the other hand, we can replace $J_1(ks_1)$ by $\frac{1}{2}ks_1$ in (7.7) to find a first approximation to $\partial\psi_1/\partial z_1$. That gives

$$\frac{\partial\psi_1}{\partial z_1}(s_1, 0) \sim \alpha_1 s_1^2, \quad \text{i.e.} \quad q_{1\theta} \sim U\delta_1^{\frac{1}{3}}\alpha_1\theta, \tag{7.9}$$

where

$$\alpha_1 = \frac{1}{2}\int_0^\infty\int_0^\infty\int_0^\infty s^2k^2 e^{-kz_1} J_1(ks)\, b_1(s^2z_1)\, dz_1\, ds\, dk.$$

We note that this expression for α_1 can be transformed, by putting $\sigma = z_1^{\frac{1}{3}}s$, $\kappa = z_1^{-\frac{1}{3}}k$, $\zeta = z_1^{\frac{2}{3}}$ and changing the order of integration, to

$$\alpha_1 = \frac{1}{3}\int_0^\infty\int_0^\infty\int_0^\infty \kappa^2 J_1(\kappa\sigma)\, e^{-\kappa\zeta}\zeta^{-\frac{1}{2}}\sigma^2 b_1(\sigma^2)\, d\kappa\, d\zeta\, d\sigma.$$

The κ integral is known (Watson 1944, p. 386, equation (6)), and we obtain finally after a little manipulation

$$\alpha_1 = \frac{3}{2}\int_0^\infty t^{\frac{2}{3}}(1+t^2)^{-\frac{5}{4}}\, dt \int_0^\infty b_1(x^3)\, dx$$
$$\doteqdot 0{\cdot}5749 \int_0^\infty b_1(x^3)\, dx, \tag{7.10}$$

for the (non-dimensional) rate of strain in the perturbation flow very near the front stagnation point.

We now consider the neighbourhood of the rear stagnation point. Obviously equations (7.1) to (7.10) all hold for the interior flow there if θ is replaced by ϕ, where

$$\phi = \pi - \theta. \tag{7.11}$$

Analogous equations hold for the exterior flow if we replace subscripts 1 by 0 (except on the Bessel functions), the function $g(z) = f_1(X_e, Y)$ by $g_0(z) = f_0(X_e, Y)$, and define z_0 by

$$r-a = a\delta_0^{\frac{1}{3}} z_0, \tag{7.12}$$

so that z_0 and z_1 are both positive in their regions of definition. This makes (7.7), (7.9) and (7.10) carry over directly to the exterior flow but causes sign changes in the analogues of (7.5) and (7.6) as follows:

$$z = -3s_0^2 z_0/4\sqrt{2}, \tag{7.13}$$

$$b_0(s_0^2 z_0) = -\frac{a^2}{U} B_0(\tfrac{3}{2}Ua^2\delta_0 s_0^2 z_0) = -\frac{a^2}{U} B_0(\overline{\psi}_0). \tag{7.14}$$

We now turn to the problem of determining the structure of the inner viscous boundary layers.

The co-ordinates X, Y are singular at the stagnation points, so we return to the dimensionless physical co-ordinates y, θ defined in (3.2), in terms of which the boundary-layer equation is (3.3). If we define a non-dimensional flux Q by

$$Q = \int_{-\infty}^{0} (u_1 - u_1^\infty)\, dy + \int_0^\infty (u_0 - u_0^\infty)\, dy,$$

we find from (3.3) that near the front stagnation point

$$\frac{3}{2}\frac{d}{d\theta}(\theta Q) + 3Q = \frac{\partial u_1}{\partial y} - \frac{\partial u_0}{\partial y}, \tag{7.15}$$

the derivatives on the right-hand side being evaluated at $y = 0$, and near the rear stagnation point

$$\frac{3}{2}\frac{d}{d\phi}(\phi Q) + 3Q = -\frac{\partial u_1}{\partial y} + \frac{\partial u_0}{\partial y}, \tag{7.16}$$

where ϕ is defined in (7.11).

The boundary conditions at $y = 0$ are

$$\delta_0 u_0 = \delta_1 u_1, \tag{7.17}$$

$$\mu_0 \frac{\partial u_0}{\partial y} = \mu_1 \frac{\partial u_1}{\partial y}, \tag{7.18}$$

since the other terms in the exact boundary conditions (3.5) and (3.6) are negligible in the stagnation region.

We assume profiles

$$u_i = u_i^\infty + A_i(\theta)\, e^{\mp y/T(\theta)},$$

where the upper sign is taken for $i = 0$ and the lower sign for $i = 1$ and where $T(\theta)$ is a dimensionless boundary-layer thickness. The functions $A_i(\theta)$ are determined by (7.17) and (7.18), and we then easily find from (7.15) and (7.16) that

$$T(\theta) = \frac{1}{|q_{1\theta}^{(0)}|}\left\{\theta^{-6}\int_0^\theta \tfrac{4}{3}[q_{1\theta}^{(0)}]^2 \theta^5\, d\theta\right\}^{\frac{1}{2}} \tag{7.19}$$

for the front inner viscous boundary layer, in which $q_{1\theta}^{(0)} = U\delta_1 u_1^\infty$, and that

$$T(\phi) = \frac{1}{|q_{1\theta}^{(0)} - q_{0\theta}^{(0)}|}\left\{\phi^{-6}\int_\phi^\infty \tfrac{4}{3}(q_{1\theta}^{(0)} - q_{0\theta}^{(0)})^2 \phi^5\, d\phi\right\}^{\frac{1}{2}} \tag{7.20}$$

for the rear inner viscous boundary layer. (It can be shown from (7.7) that

$$q_{1\theta}^{(0)} - q_{0\theta}^{(0)} = O\left(\frac{U \ln \phi}{R\phi^4}\right)$$

for $\phi \gg R^{-\frac{1}{3}}$, so that the integral in (7.20) is convergent, and we incur negligible errors in the region $\phi = O(R^{-\frac{1}{3}})$ by writing its upper limit as ∞.) We note from (7.9) that $T(\theta)$ is well behaved and $\to 1/\sqrt{6}$ as $\theta \to 0$ while $T(\phi) \to \infty$ as $\phi \to 0$, so that the rear inner viscous boundary layer is itself singular at the rear stagnation point.

We will use this approximate solution in §8 to estimate the dissipation.

8. The second-order terms in the drag

We are now in a position to calculate the second-order terms in the expansion of the drag coefficient for large Reynolds numbers. We use the method given in DM which involves calculating the viscous dissipation in the boundary layers and wakes. Equation (4.18) of DM, originally derived for a region in which there are small perturbations to an irrotational flow, is easily shown to be true also for small perturbations to a Hill's spherical vortex (or any other axisymmetric flow with closed streamlines, obeying Batchelor's circulation condition and bounded by a stream-surface).

Using this method, we find for the drag coefficient C_D

$$C_D = \frac{48}{R_0}\left\{1 + \frac{3}{2V} + \frac{1}{R_0^{\frac{1}{2}}}\left[\frac{3}{4}\int_{-\infty}^{0} \{f_0(-1, z)\}^2 dz \right.\right.$$
$$+ 3\sqrt{2}C\frac{V'}{V^2} + \sqrt{2}\int_{-1}^{+1} f_0(\mu, 0)\, d\mu + \frac{3V'}{\sqrt{2}\,V^2}\int_{-1}^{+1} f_1(\mu, 0)\, d\mu$$
$$\left.\left. + \frac{1}{8}\int_{-\infty}^{0}\int_{-1}^{+1}\left(\frac{\partial f_0}{\partial z}\right)^2 d\mu\, dz + \frac{1}{8}\frac{V'}{V^2}\int_{0}^{\infty}\int_{-1}^{+1}\left(\frac{\partial f_1}{\partial z}\right)^2 d\mu\, dz\right]\right\}, \tag{8.1}$$

where the f_i are written as functions of $\mu = \cos\theta$ and z, which is $-3(r-a)/(4a\delta_i\sqrt{2})$, from (3.2), (3.8) and (5.1).

The first term in the square brackets is the wake dissipation. We saw in §6 that the inner viscous boundary layers do not affect the matching procedure which determines the wake structure, so that this expression for the wake dissipation (which is proved in DM) is valid, although, strictly, the integrand should be $\{f_0(-1+\eta R_0^{-\frac{1}{3}}, z)\}^2$ where $1 \ll \eta \ll R_0^{\frac{1}{3}}$. This is because the matching is performed at a value of θ where the boundary-layer solution of §3 is valid (figure 2a); however, the error introduced in C_D by evaluating f_0 at $\mu = -1$ is of smaller order than $O(R_0^{-\frac{3}{2}})$.

The second term is the dissipation associated with the extra circulation in the Hill's spherical vortex and it presents no problems.

It is the μ-integrations which cause trouble. Let us first note that the f_i are continuous and $(\partial f_i/\partial z)^2$ integrable at $\mu = -1$, while as $\mu \to 1$, if $z \to 0$ in such a way that z^2/X is finite,

$$f_0 \sim \frac{V' g(0)}{V(1+V')}\,\mathrm{erfc}\left[\frac{-2z}{(1-\mu)\sqrt{3}}\right], \tag{8.2}$$

and
$$f_1 \sim g(0)\left\{1-\frac{V'}{V'+1}\,\mathrm{erfc}\left[\frac{2z}{(1-\mu)\sqrt{3}}\right]\right\}. \tag{8.3}$$

Thus the integrals of f_0 and f_1 converge at $\mu = 1$, while those of $(\partial f_0/\partial z)^2$ and $(\partial f_1/\partial z)^2$ diverge there.

Now the error in f_0 or f_1 due to the failure of the theory of § 3 is $O(1)$ and is confined to small ranges $-1 < \mu < -1+O(R^{-\frac{1}{3}})$ and $1-O(R^{-\frac{1}{3}}) < \mu < 1$. Thus since the integrals of f_0, f_0^2 and f_1 are well behaved at $\mu = \pm 1$, we can neglect the corrections to f_0 and f_1 introduced by the inner viscous boundary layers when these integrals are being computed. It remains only to consider the integrals of $(\partial f_i/\partial z)^2$.

The theory of § 3 gives the order of $(\partial f_i/\partial z)$ to be unity when $\mu \to -1$. This is an underestimate and is caused by the underestimate of normal derivatives implicit in the neglect of the inner viscous boundary layers. Actually, from the discussion of the inner viscous boundary layers, $(\partial f_i/\partial y)$ is $O(1)$ there, and, since

$$dz = -\tfrac{1}{2}(X_e)^{-\frac{1}{2}}\phi^2 dy,$$

it follows that $\partial f_i/\partial z$ is $O(R^{\frac{1}{3}})$ in the rear stagnation region. Thus the contribution to $\iint (\partial f_i/\partial z)^2 d\mu\, dz$ from the range $-1 < \mu < -1+O(R^{-\frac{1}{3}})$ is of order

$$R^{\frac{2}{3}}R^{-\frac{1}{3}}R^{-\frac{1}{3}} = 1$$

so that, as we saw in § 6, we must allow for the dissipation in the rear inner viscous boundary layer. In view of the above discussion, we can do this to the required order of accuracy by integrating from $\mu = -1$ with the explicit formulae for f_i, (3.19) and (3.20), and simply adding on the dissipation from the rear inner viscous boundary layer.

In the front stagnation region, we see from (8.2) and (8.3) that the boundary-layer theory of § 3 gives $\partial f_i/\partial z$ to be $O(R^{+\frac{1}{3}})$, but makes $\partial f_i/\partial z \to \infty$ as $\mu \to 1$, whereas the correct derivatives are $O(R^{\frac{1}{3}})$ and *bounded*. This spurious infinity is due to the approximation to the starting profile which is strictly valid only if $R = \infty$. (See §§ 6 and 7 for a discussion.) Thus we can employ (3.19) and (3.20) only up to $\mu = 1-\eta R_0^{-\frac{1}{3}}$, where $1 \ll \eta \ll R_0^{\frac{1}{3}}$, and we must evaluate the contributions from the range $1 > \mu > 1-\eta R_0^{-\frac{1}{3}}$ from the solution of § 7 for the inner viscous boundary layer.

Now from (8.2) and (8.3)

$$\frac{1}{8}\int_{-1}^{1-\eta R_0^{-\frac{1}{3}}}\int_{-\infty}^{0}\left(\frac{\partial f_0}{\partial z}\right)^2 dz\,d\mu+\frac{V'}{8V^2}\int_{-1}^{1-\eta R_0^{-\frac{1}{3}}}\int_{0}^{\infty}\left(\frac{\partial f_1}{\partial z}\right)^2 dz\,d\mu$$
$$= \frac{\ln R_0}{6\sqrt{(6\pi)}\,(1+V')}\left\{\frac{V'g(0)}{V}\right\}^2+O(1), \tag{8.4}$$

so that, as anticipated in § 6, there is a logarithmic term in the correction to the first-order drag. However, the contribution of this term is rather small for the Reynolds numbers which occur in practice. (The highest R_0 values for approximately spherical drops are only a few hundred, so that $\ln R_0 < 7$.)

The contribution to the term in square brackets in (8.1) for the dissipation in the front inner viscous boundary layer can be approximated by using the momentum integral solution, as

$$\frac{1}{6\sqrt{2}(1+V')\delta_0^2}\int_0^{\theta^*}\frac{\theta(q_{1\theta}^{(0)}/U)^2\,d\theta}{T(\theta)}, \tag{8.5}$$

where $T(\theta)$ is defined in (7.17), and where θ^* is $O(R^{-\frac{1}{6}})$. We can estimate the integral by using an approximation to $q_{1\theta}^{(0)}$. For all $\theta \leqslant \theta^*$ we replace $q_{1\theta}^{(0)}$ by $U\delta_1^{\frac{1}{2}}\alpha_1\theta$,

$V'\ldots$	0·2	0·5	1·0	2·0	5·0	∞
$3\lambda_a\ldots$	2	1	0	−1	−2	−3
c_1	0·0275	0·0935	0·177	0·262	0·335	0·390
c_2	8·89	7·56	6·15	5·15	4·22	3·56
c_3	−9·72	−9·00	−8·22	−7·41	−6·59	−5·77

TABLE 3. The second-order terms in equation (8.7) for the drag coefficient.

which by (7.9) is exactly valid only for $\theta \ll \theta^*$, and we require that, at $\theta = \theta^*$, this linear approximation to $q_{1\theta}^{(0)}$ agrees with the asymptotic form (7.8). Thus we get

$$U\delta_1^{\frac{1}{2}}\alpha_1\theta^* = U\delta_1 g(0)/\theta^*,$$

which fixes θ^*. Using this approximation the contribution to the square bracket in (8.1) is approximately

$$\left\{\frac{V'g(0)}{V}\right\}^2\frac{1}{8\sqrt{3}(1+V')}, \tag{8.6}$$

which is numerically fairly unimportant, being $1{\cdot}88/\ln R_0$ times the logarithmic contribution (8.4), or less than 5 % of the terms from the wake and the principal boundary layer.

A similar approximation can be used for the rear inner viscous boundary layer and the dissipation there also can be shown to be small.

We neglected these two contributions when calculating (8.7) below, which greatly shortens the work, though it means that our results for C_D will be slightly too small, by an amount of the same order as the logarithmic term for Reynolds numbers up to a few hundred.

On performing the various integrations numerically and collecting terms in a form suitable for interpolation, we find that

$$C_D = \frac{48}{R_0}\left\{1+\frac{3}{2V}+\frac{\ln R_0}{R_0^{\frac{1}{2}}}\frac{\lambda_b^2(1+V')}{V^2}c_1 + \frac{\lambda_b(1+V')}{R_0^{\frac{1}{2}}V}\left[\frac{\lambda_b(1+V')}{V}c_2+\left(1+\frac{3}{2V}\right)c_3\right]\right\}, \tag{8.7}$$

where c_1, c_2, c_3 are the functions of λ_a (or of V') given in table 3. When $V = V' = \infty$, we recover the result of DM that $C_D = 48R_0^{-1}(1-2{\cdot}21R_0^{-\frac{1}{2}})$. The errors are at most 5 % with the steplengths of 0·2 in both z and μ which were used for numerical integration; improving the results would have required excessive computer time as well as demanding a more detailed analysis of the motion near the stagnation points which would have involved tabulating functions of two variables (V and

λ_a) instead of one (λ_a). The present accuracy is adequate for comparison with the experiments, whose errors are of the same order, and this is the subject of the next section.

9. Comparison with experimental results

For a fair test of our theory, it is obviously necessary to compare it with experiments on drops satisfying reasonably well the assumptions on which the theory is

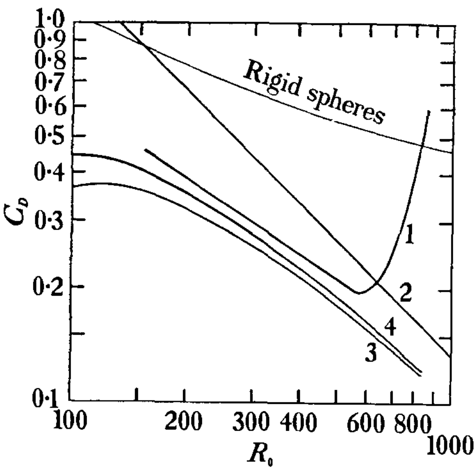

FIGURE 3

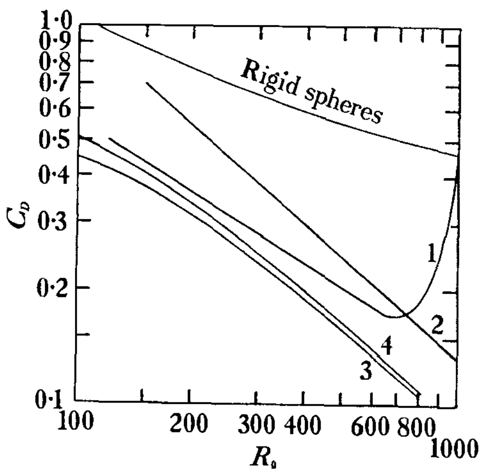

FIGURE 4

FIGURE 3. Bromobenzene drops in water. 1, experiments (Winnikow & Chao 1966); 2, first-order theory; 3, second-order theory without the logarithmic term; 4, second-order theory with the logarithmic term.

FIGURE 4. Chlorobenzene drops in water. 1, experiments (Winnikow & Chao 1966); 2, first-order theory; 3, second-order theory without the logarithmic term; 4, second-order theory with the logarithmic term.

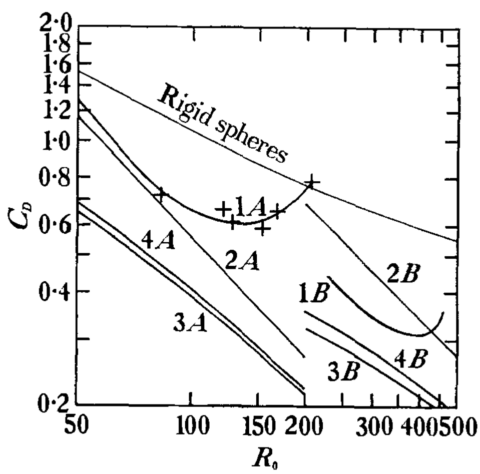

FIGURE 5. A, water drops in Finol (Elzinga & Banchero 1961); B, ethyl chloroacetate drops in water (Licht & Narasimhamurty 1955). 1, experiments; 2, first-order theory; 3, second-order theory without the logarithmic term; 4, second-order theory with the logarithmic term.

based. The most important points are that the interface must be uncontaminated, and that the Reynolds number be both high enough for perturbation quantities to be small and low enough for the drop to be nearly spherical. Contamination is clearly present if the experimental drag coefficients are close to those of rigid spheres over a range of Reynolds numbers, and a graph of C_D against R_0 gives an easy and quick check on this. The other criterion is only a little more complicated; we take the upper limit of R_0 for nearly spherical behaviour to be that of the experimental minimum value of C_D. This seems justified because Winnikow & Chao (1966) showed in their figure 4 that instability does not set in until still higher Reynolds numbers for drops, and Moore (1965) showed that, for gas bubbles in steady flow, the minimum and subsequent rapid rise of C_D with R_0 can be explained in terms of a gradual transition from spherical to spheroidal shape. We assume merely that Moore's explanation of the minimum holds also for drops. Having found this upper limit of R_0, we predict the corresponding value of the internal circulation from (5.15). Our theory should then hold if the circulation factor $(1+C\delta_1)$ at this value of R_0 is close to unity, and it is clearly not applicable if it turns out to be small or negative. That would mean that the 'small' perturbation quantities were of the same order as, or greater than, those of the basic flow.

On examining in this way the experimental results of Hu & Kintner (1955), Licht & Narasimhamurty (1955), Klee & Treybal (1956), Keith & Hixson (1955), Elzinga & Banchero (1961) and Winnikow & Chao (1966), we found only four liquid combinations with satisfactory purity and with $1+C\delta_1$ greater than $\frac{1}{2}$ at minimum drag coefficient, namely Licht & Narasimhamurty's drops of ethyl chloroacetate in water, Elzinga & Banchero's drops of water in Finol, and Winnikow & Chao's drops of bromobenzene and chlorobenzene in water. The respective values of $1+C\delta_1$ at minimum drag coefficient were 0·58, 0·90, 0·65, 0·75. As V and V' were not given separately by Winnikow & Chao, but only the combination $b = (2+3/V)(1+1/V')$, values of V and V' were calculated by assuming the densities of the liquids to be those given in the *Handbook of Chemistry and Physics* (Hodgman 1960), and using $V' = (\rho_0 V/\rho_1)^{\frac{1}{2}}$ to obtain a quadratic equation for $V^{\frac{1}{2}}$. The results are given in table 4, together with the predictions for $(1+C\delta_1)$ and C_D as functions of R_0, calculated for all four cases mentioned above from (5.15) and (8.7).

As none of the experimenters mentioned above reported measurements of the internal circulation, we cannot compare its value directly with that of $(1+C\delta_1)$ given by our theory. Drag coefficients can be compared, as follows: figures 3–5 show the experimental (curve 1) and theoretical (curves 2–4 for different orders of approximation) values of C_D as functions of R_0, plotted logarithmically on both axes.

For both of Winnikow & Chao's liquids the general trend of C_D with R_0 is reasonably well shown in the range $150 < R_0 < 500$, although the numerical values from our best approximation (curve 4) are from 6 to 20% too low. This is perhaps to be expected: the terms of order $R_0^{-\frac{3}{2}}$ in the theory are only approximations, whose error is of the order of the distance between curves 3 and 4; higher terms have not been included at all, nor have the effects of distortion from a spherical shape. The agreement of the theory with the other experiments is rather worse, but their authors did not report such elaborate precautions against impurities in their liquids as Winnikow & Chao, and the Reynolds numbers were not so high.

	b	ρ_0/ρ_1	V	V'	$1+C\delta_1$	C_D
Bromobenzene drops in water	2·38	0·672	0·847	0·753	$1-8{\cdot}4R_0^{-\frac{1}{2}}$	$133R_0^{-1}(1-10{\cdot}1R_0^{-\frac{1}{2}}+0{\cdot}135R_0^{-\frac{1}{2}}\ln R_0)$
Chlorobenzene drops in water	2·27	0·903	1·25	1·06	$1-6{\cdot}6R_0^{-\frac{1}{2}}$	$106R_0^{-1}(1-8{\cdot}5R_0^{-\frac{1}{2}}+0{\cdot}126R_0^{-\frac{1}{2}}\ln R_0)$
Ethyl chloro-acetate drops in water	2·60	0·871	0·812	0·840	$1-8{\cdot}6R_0^{-\frac{1}{2}}$	$137R_0^{-1}(1-6{\cdot}74R_0^{-\frac{1}{2}}+0{\cdot}138R_0^{-\frac{1}{2}}\ln R_0)$
Water drops in Finol	1·71	0·831	11·8	3·13	$1-1{\cdot}1R_0^{-\frac{1}{2}}$	$54R_0^{-1}(1-2{\cdot}64R_0^{-\frac{1}{2}}+0{\cdot}072R_0^{-\frac{1}{2}}\ln R_0)$

TABLE 4. Dimensionless groups and theoretical predictions for the experimental pairs of fluids

We conclude that the theory is a valid approximation for drops in liquids which satisfy the conditions of small viscosity both inside and outside the drop, large interfacial tension, and high purity.

Appendix

To solve the diffusion equation (3.11) with its given boundary and initial conditions, we first find the effects of: (i) the discontinuity of $\mu_i\,\partial f_i/\partial Y$ given by (3.13), assuming that $\delta_i f_i$ is continuous across $Y = 0$ and zero at $X = 0$; (ii) the discontinuity of $\delta_i f_i$ given by (3.12), assuming that $\mu_i\,\partial f_i/\partial Y$ is continuous across $Y = 0$ and $f_i = 0$ at $X = 0$; (iii) the initial values of $\delta_i f_i$ at $X = 0$ given by (3.17) and (3.18), assuming that $\delta_i f_i$ and $\mu_i\,\partial f_i/\partial Y$ are both continuous across $Y = 0$. We then add the results of these three calculations, obtaining the value of $f_i(X, Y)$ for the actual problem, because all the governing equations are linear.

(i) *The discontinuity of* $\mu_i\,\partial f/\partial Y$

This problem is formally so similar to that of DM (equations (2.28) to (2.31)) that its solution must be some multiple of that given by DM, both inside and outside the drop. We therefore put

$$f_i(X, Y) = \alpha_i X^{\frac{1}{2}}\phi_i(Y/2X^{\frac{1}{2}}), \tag{A1}$$

where

$$\phi_0(t) = \phi(t) = \pi^{-\frac{1}{2}}e^{-t^2} - t\operatorname{erfc} t,$$

$$\phi_1(t) = \phi(-t) = \pi^{-\frac{1}{2}}e^{-t^2} + t\operatorname{erfc}(-t).$$

The boundary conditions then give

$$\delta_0\alpha_0 = \delta_1\alpha_1 = \frac{(9\mu_1+6\mu_0)\delta_0\delta_1}{\mu_1\delta_0+\mu_0\delta_1}. \tag{A2}$$

(ii) *The discontinuity of* $\delta_i f_i(X, Y)$

If $k_i(X, Y) = \mu_i f_i(X, Y)$ the boundary conditions are that $k_i(0, Y) = 0$ and $\partial k_i/\partial Y$ is continuous across $Y = 0$, together with

$$\frac{\delta_0}{\mu_0}k_0 - \frac{\delta_1}{\mu_1}k_1 = \tfrac{3}{2}\delta_1 C\sin^2\theta(X) \tag{A3}$$

at $Y = 0$. Now consider the function $K(X, Y)$ which satisfies $K_X = K_{YY}$, with boundary conditions given by $K(X, 0) = \sin^2\theta(X)$ and $K(0, Y) = 0$. By Carslaw & Jaeger (1959, §2.5)

$$K(X, Y) = N(X, |Y|), \tag{A 4}$$

as defined in (3.22). Therefore

$$\mu_i f_i(X, Y) = k_i(X, Y) = (-1)^i \tau N(X, Y), \tag{A 5}$$

where, from (A 3),

$$\tau = \frac{3\delta_1\mu_0\mu_1 C}{2(\delta_1\mu_0 + \delta_0\mu_1)}. \tag{A 6}$$

(iii) *The initial-value problem*

We now have to solve $f_{iX} = f_{iYY}$ with the conditions

$$\left.\begin{aligned} f_1(0, Y) &= \gamma(Y) \quad \text{for} \quad Y < 0, \\ f_0(0, Y) &= 0 \quad \text{for} \quad Y > 0, \end{aligned}\right\} \tag{A 7}$$

where

$$\left.\begin{aligned} \delta_1 f_1(X, 0) &= \delta_0 f_0(X, 0), \\ \mu_1 f_{1Y}(X, 0) &= \mu_0 f_{0Y}(X, 0). \end{aligned}\right\} \tag{A 8}$$

Define

$$\left.\begin{aligned} g(X, Y) &= \delta_0 f_0(X, Y) - \delta_1 f_1(X, -Y) \quad \text{for} \quad Y > 0, \\ &= \delta_1 f_1(X, Y) - \delta_0 f_0(X, -Y) \quad \text{for} \quad Y < 0. \end{aligned}\right\} \tag{A 9}$$

Then $g(X, Y)$ is an odd function of Y which obeys the diffusion equation and vanishes at $Y = 0$, so that it is regular for all $X > 0$, and at $X = 0$

$$g(0, Y) = \pm\delta_1\gamma(\pm Y) \quad \text{according as } Y \lesseqgtr 0.$$

It is easy to see now (Carslaw & Jaeger 1959, §2.2) that

$$g(X, Y) = \frac{\delta_1}{2(\pi X)^{\frac{1}{2}}}\int_{-\infty}^{0}\gamma(Y')\left\{\exp\left[-\frac{(Y-Y')^2}{4X}\right] - \exp\left[-\frac{(Y+Y')^2}{4X}\right]\right\}dY'. \tag{A 10}$$

Also, let

$$\left.\begin{aligned} h(X, Y) &= \mu_0 f_0(X, Y) + \mu_1 f_1(X, -Y) \quad \text{for} \quad Y > 0, \\ &= \mu_1 f_1(X, Y) + \mu_0 f_0(X, -Y) \quad \text{for} \quad Y < 0. \end{aligned}\right\} \tag{A 11}$$

Then $h(X, Y)$ is an even function of Y whose derivative h_Y vanishes at $Y = 0$ and which, like $g(X, Y)$, obeys the diffusion equation and must be regular for $X > 0$, while

$$h(0, Y) = \mu_1\gamma(-|Y|). \tag{A 12}$$

The analogue of (A 10) for this case is

$$h(X, Y) = \frac{\mu_1}{2(\pi X)^{\frac{1}{2}}}\int_{-\infty}^{0}\gamma(Y')\left\{\exp\left[-\frac{(Y-Y')^2}{4X}\right] + \exp\left[-\frac{(Y+Y')^2}{4X}\right]\right\}dY'. \tag{A13}$$

Finally, $f_i(X, Y)$ may be obtained by solving (A 9) and (A 11), as

$$f_i(X, Y) = \frac{\mu_0\mu_1}{\mu_i(\mu_0\delta_1 + \mu_1\delta_0)}g(X, Y) + \frac{\delta_0\delta_1}{\delta_i(\mu_0 + \mu_1)}h(X, Y). \tag{A 14}$$

It now remains only to add the contributions to f_1 given by (A 1), (A 5) and

(A 14), to obtain equations (3.19) and (3.20). In terms of the variables of §5, which are more convenient for numerical calculations,

$$f_0 = \frac{\lambda_b V'}{(V'+1)V}\left\{-(6V+9)X^{\frac{1}{2}}\phi\left(-\frac{X_e^{\frac{1}{2}}z}{X^{\frac{1}{2}}}\right)+\frac{3}{2}\frac{C}{\lambda_b}N(X,\,-2X_e^{\frac{1}{2}}z)\right.$$
$$\left.+\frac{2X_e^{\frac{1}{2}}}{(\pi X)^{\frac{1}{2}}}\int_0^\infty \frac{g(z')}{\lambda_b}\exp\{-X_e(z-z')^2/X\}dz'\right\}, \tag{A 15}$$

$$f_1 = \lambda_b\left\{\frac{(6V'+9)}{V'+1}X^{\frac{1}{2}}\phi\left(\frac{X_e^{\frac{1}{2}}z}{X^{\frac{1}{2}}}\right)-\frac{3}{2}\frac{C}{\lambda_b}\frac{V'}{V'+1}N(X,2X_e^{\frac{1}{2}}z)\right.$$
$$+\frac{X_e^{\frac{1}{2}}}{(\pi X)^{\frac{1}{2}}}\int_0^\infty \frac{g(z')}{\lambda_b}\exp\{-X_e(z-z')^2/X\}dz'$$
$$\left.+\frac{X_e^{\frac{1}{2}}}{(\pi X)^{\frac{1}{2}}}\lambda_a\int_0^\infty \frac{g(z')}{\lambda_b}\exp\{-X_e(z+z')^2/X\}dz'\right\}. \tag{A 16}$$

REFERENCES

Batchelor, G. K. 1956 *J. Fluid Mech.* **1**, 177–90.

Burggraf, O. R. 1966 *J. Fluid Mech.* **24**, 113–51.

Carslaw, H. S. & Jaeger, J. C. 1959 *Conduction of Heat in Solids*, 2nd ed. Oxford University Press.

Chao, B. T. 1962 *Phys. Fluids*, **5**, 69–79.

Conkie, W. R. & Savic, P. 1953 *Nat. Res. Coun. Can., Rept.* no. MT-23.

Elzinga, E. R. & Banchero, J. T. 1961 *A.I.Ch.E. J.* **7**, 394–9.

Garner, F. H. & Haycock, P. J. 1959 *Proc. Roy. Soc.* A **252**, 457–75.

Harper, J. F., Moore, D. W. & Pearson, J. R. A. 1967 *J. Fluid Mech.* **27**, 361–6.

Harper, J. F. 1963 *J. Fluid Mech.* **17**, 141–53.

Hill, M. J. M. 1894 *Phil. Trans. Roy. Soc.* **185**, 213–45.

Hodgman, C. D. 1960 *Handbook of Chemistry and Physics*, 42nd ed. Chemical Rubber Publ. Co., Cleveland, Ohio.

Hu, S. & Kinter, R. C. 1955 *A.I.Ch.E. J.* **1**, 42–8.

Keith, F. W. & Hixson, A. N. 1955 *Ind. Engng Chem.* **47**, 258–67.

Klee, A. J. & Treybal, R. E. 1956 *A.I.Ch.E. J.* **2**, 444–7.

Lamb, H. 1932 *Hydrodyamics*, 6th ed. Cambridge University Press.

Levich, V. G. 1949 *Zhur. Exsp. I. Teoret. Fiz.* **23**, 71.

Licht, W. & Narasimhamurty, G. S. R. 1955 *A.I.Ch.E. J.* **1**, 366–73.

Mills, R. D. 1965 *J. Roy. Aeronaut. Soc.* **69**, 116–20.

Moore, D. W. 1959 *J. Fluid Mech.* **6**, 113–30.

Moore, D. W. 1963 *J. Fluid Mech.* **16**, 161–76.

Moore, D. W. 1965 *J. Fluid Mech.* **23**, 747–66.

Watson, G. N. 1944 *Theory of Bessel Functions*, 2nd ed. Cambridge University Press.

Winnikow, S. & Chao, B. T. 1966 *Phys. Fluids*, **9**, 50–61.

[*Note added in proof.*] Professor K. Stewartson has recently discussed the integral equation (5.11) and an account of his work is to appear shortly.